

The logarithmic residue density of a generalised Laplacian

Jouko Mickelsson and Sylvie Paycha

August 19, 2010

Abstract

We show that the residue density of the logarithm of a generalised Laplacian on a closed manifold defines an invariant polynomial valued differential form. We express it in terms of a finite sum of residues of classical pseudodifferential symbols. In the case of the square of a Dirac operator, these formulae provide a pedestrian proof of the Atiyah-Singer formula for a pure Dirac operator in dimension 4 and for a twisted Dirac operator on a flat space of any dimension. These correspond to special cases of a more general formula by S. Scott and D. Zagier announced in [Sc2] and to appear in [Sc3]. In our approach, which is of perturbative nature, we use either a Campbell-Hausdorff formula derived by Okikolu or a non commutative Taylor type formula.

Introduction

The noncommutative residue on classical pseudodifferential operators has a notable property, locality i.e., it corresponds to a residue density $\text{res}_x(A) dx$ integrated over an n -dimensional (closed) manifold M :

$$\text{res}(A) = \int_M \text{res}_x(A) dx; \quad \text{res}_x(A) dx := \frac{1}{(2\pi)^n} \int_{|\xi|=1} \text{tr}(\sigma_{-n}(A)(x, \xi)) d\xi, \quad (1)$$

where $\sigma(A)(x, \xi)$ stands for the local symbol of A , $\sigma_{-n}(A)(x, \xi)$ for the $-n$ -th homogeneous part of its symbol, with (x, ξ) varying in the cotangent bundle of M and where tr stands for the fibrewise trace. As it was observed in [O2], this extends to logarithms $A = \log Q$ of an elliptic pseudodifferential operator Q of positive order with appropriate spectral cut (we call it admissible).

Exponentiating $\text{res}(\log(Q))$ leads to the residue determinant $\det_{\text{res}}(Q) := e^{\text{res}(\log Q)}$ first introduced by Wodzicki in the case of zero order operators (see e.g. the survey [Ka]) and further extended by Scott [Sc1] to elliptic pseudodifferential operator Q with appropriate spectral cut of positive order. Further logarithmic structures were since then investigated in [Sc2] in relation with topological quantum field theory. Here, we show that the logarithmic residue density for a generalised Laplacian Q ,

$$\text{res}_x(\log Q) dx := \frac{1}{(2\pi)^n} \left[\int_{S_x^* M} \text{tr}(\sigma_{-n}(\log Q)(x, \xi)) d\xi \right] dx,$$

which defines an invariant polynomial valued form in the sense of Weyl (Theorem 1). It then follows from Gilkey's invariance theory, that it can be expressed in terms of Pontryagin and Chern classes.

The presence of a logarithm makes the actual computation of a logarithmic residue density difficult. However, observing that the symbol of a generalised Laplacian reads $\sigma(Q) = |\xi|^2 + \sigma_{<2}(Q)$ where $\sigma_{<2}(Q)$ is of order smaller than 2 enables us to carry out computations by means of a noncommutative Taylor type formula (Theorem 3) or a Campbell-Hausdorff formula (Theorem 2), both of which provide ways to compare $\sigma(\log Q)$ with $\log(|\xi|^2)$.

A similar procedure applies on the operator level to compute the integrated logarithmic density $\int_M \text{res}_x(\sigma(\log Q)) dx$ in the special case of the square of a twisted Dirac operator D_W acting on a twisted \mathbb{Z}_2 -graded spinor bundle $E = S \otimes W$. Indeed, combining the Lichnerowicz formula (36) (which compares D_W^2 with a Laplace-Beltrami operator $\Delta^E = (\nabla^E)^* \nabla^E$ built from the underlying connection ∇^E on E) with a Campbell-Hausdorff formula (which compares $\log D_W^2$ with $\log \Delta^E$) yields an expression for the integrated logarithmic (super-) residue density $\text{sres}(\log D_W^2)$ in terms of $\text{sres}(\log(\Delta^E))$ and a finite number of (super-) residues of classical operators involving the curvature of ∇^E (Theorem 5). This integrated logarithmic (super-) residue density turns out to be proportional to the index of the chiral Dirac operator D_W^{+1} (Theorem 4):

$$\text{ind}(D^+) = -\frac{1}{2} \int_M \text{sres}_x(\log D_W^2) dx. \quad (2)$$

Thus, locality in the Atiyah-Singer index theorem is closely related to the local property of the noncommutative residue.

We compute the index in two concrete examples, first for a twisted Dirac operator on a flat space (Theorem 6) along the lines described above using a Campbell-Hausdorff formula, then for a pure Dirac operator in dimension four using a Taylor type formula. For the second example we first derive simple formulae (see Proposition 5) for (super-) residues of certain expressions involving the derivatives of the Christoffel symbols, which can then be used to derive the index in dimension four. We recover this way, the Atiyah-Singer index theorem for a pure Dirac operator on a four dimensional spin manifold.

With the perturbative approach adopted here using either a Campbell-Hausdorff or a noncommutative Taylor formula, we were unfortunately unable to derive the general Atiyah-Singer formula announced in [Sc2] and to appear [Sc3]. This perturbative approach nevertheless provides a pedestrian proof in the cases investigated here and useful intermediate results such as Theorem 1 and Theorem 3, which we find are of interest in their own right.

Notations

Given an even $n = 2p$ -dimensional real oriented euclidean vector space V , there is a unique \mathbb{Z}_2 -graded complex Clifford module $S = S^+ \oplus S^-$, the spinor module, such that the complex Clifford algebra $C(V) \otimes \mathbb{C} \simeq \text{End}(S)$ and $\dim(S) = 2^p$. An auxiliary linear complex space W yields a \mathbb{Z}_2 -graded twisted Clifford module $E = S \otimes W$. Let

$$c : \Lambda V \rightarrow C(V)$$

¹This was observed independently by S. Scott in [Sc1] and the second author in some unpublished lecture notes delivered in Colombia.

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto c(e_{i_1}) \cdots c(e_{i_k})$$

be the quantisation map. To simplify notations we set $\gamma_j = c(e_j)$, so that the grading operator reads $\Gamma = i^p \gamma_1 \cdots \gamma_n$. Notice that $\Gamma^2 = Id$. The cyclicity of the trace combined with the Clifford relations imply that the supertrace $\text{str} := \text{tr} \circ \Gamma$ on $\text{End}(E)$ satisfies the following property for a matrix $M \in \text{End}(W)$ viewed as an element of $\text{End}(E)$:

$$\text{str}(M \gamma_{i_1} \cdots \gamma_{i_k}) = 0 \quad \text{if} \quad k < n, \quad \text{str}(M \gamma_1 \cdots \gamma_n) = (-2i)^p \text{tr}(M), \quad (3)$$

since $\dim \text{End}(S) = 2^p$. On the other hand, setting

$$\sigma_{ij} = \frac{1}{8} [\gamma_i, \gamma_j] = \frac{1}{4} \gamma_i \gamma_j \quad \text{if} \quad i \neq j, \quad (4)$$

we have that for any permutation $\tau \in \Sigma_n$ with signature $|\tau|$:

$$\text{str}(\sigma_{\tau(1)\tau(2)} \sigma_{\tau(3)\tau(4)} \cdots \sigma_{\tau(n-1)\tau(n)}) = \frac{(-1)^{|\tau|}}{4^p} \text{str}(\gamma_1 \cdots \gamma_n) = \frac{(-1)^{|\tau|} (-i)^p}{2^p}. \quad (5)$$

These constructions carry out to bundles, for which we abusively use the same notations. Let $E = S \otimes W = E^+ \oplus E^-$, with $E^+ = S^+ \otimes W$, $E^- = S^- \otimes W$ be a twisted \mathbb{Z}_2 -graded spinor bundle over an even $n = 2p$ -dimensional closed Riemannian manifold M , with auxillary bundle W equipped with a connection ∇^W .

Let, for a vector bundle F over M , $C\ell(M, F)$ denote the algebra of classical pseudodifferential operators acting on the space $C^\infty(M, F)$ of smooth sections of the vector bundle F .

Let $D = \sum_{i=1}^n c(e_i) \nabla_{e_i}^S \in C\ell(M, S)$ be the Dirac operator, where ∇^S is the spinor connection, where c stands for the Clifford multiplication and $\{e_i, i = 1, \dots, n\}$ for an orthonormal tangent frame on M . In local coordinates we shall also write γ_i for $c(e_i)$.

Let $\nabla^E := \nabla^S \otimes 1 + 1 \otimes \nabla^W$ be a connection on the twisted bundle $E = S \otimes W$ and let $D_W = \sum_{i=1}^n c(e_i) \nabla_{e_i}^E \in C\ell(M, E)$ be the corresponding twisted Dirac operator. The chiral Dirac operators D_W^+ and its formal adjoint D_W^- act from $C^\infty(M, E^+)$ to $C^\infty(M, E^-)$ and conversely.

1 The logarithmic residue density as an invariant polynomial

Following Gilkey's notations (see (2.4.3) in [G]), for a multi-index $\alpha = (\alpha_1 \dots \alpha_s)$ we introduce formal variables $g_{ij/\alpha} = \partial_\alpha g_{ij}$ for the partial derivatives of the metric tensor g on M and the connection ω on the external bundle. Let us set

$$\text{ord}(g_{ij/\alpha}) = |\alpha| = \alpha_1 + \dots + \alpha_s; \quad \text{ord}(\omega_{i/\beta}) = |\beta|.$$

Inspired by Gilkey (see (1.8.18) and (1.8.19) in [G]), we set the following definition.

Definition 1 *We call a classical operator $A \in C\ell(M, E)$ of order a **geometric**, if in any local trivialisation, the homogeneous components $\sigma_{a-j}(A)$ are homogeneous of order j in the jets of the metric and of the connection.*

Remark 1 A differential operator $A = \sum_{|\alpha| \leq a} c_\alpha(x) \partial_x^\alpha \in C\ell(M, E)$ is geometric if $c_\alpha(x)$ is homogeneous of order $j = a - |\alpha|$ in the jets of the metric and of the connection ∇^W . Here we use the standard notation $\partial_x^\alpha = \partial_{\alpha_1} \dots \partial_{\alpha_s}$.

Example 1 The Laplace Beltrami operator

$$\Delta_g = -\frac{1}{\sqrt{g}} \sum_{i=1, j=1}^n \partial_i (\sqrt{g} g^{ij} \partial_j)$$

has this property.

More generally, formula (2.4.22) in [G] shows that $\Delta_p = d_{p-1} \delta_{p-1} + \delta_p d_p$ on p -forms, where $\delta_k = (-1)^{n_{k+1}} \star_{n-k} d_{n-k-1} \star_{k+1}$, is a geometric operator. Indeed, each derivative applied to \star reduces the order of differentiation by 1 and increases the order in the jets of the metric by 1.

Example 2 The square of the twisted Dirac operator

$$D_W^2 = - \sum_{ij} g^{ij} \left(\nabla_i^E \nabla_j^E + \sum_k \Gamma_{ij}^k \nabla_k^E \right) + \sum_{i < j} c(dx^i) c(dx^j) [\nabla_i^E, \nabla_j^E]$$

has this property.

Geometric operators form an algebra.

Lemma 1 The product of two geometric operators A and B in $C\ell(M, E)$ is again a geometric operator.

Proof: Since the product AB has symbol

$$\sigma(AB) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma(A) \partial_x^\alpha \sigma(B),$$

we have

$$\sigma_{a+b-k}(AB) = \sum_{|\alpha|+i+j=k} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_{a-i}(A) \partial_x^\alpha \sigma_{b-j}(B)$$

where a is the order of A , b the order of B . Thus, if $\sigma_{a-i}(A)$ and $\sigma_{b-j}(B)$ are homogeneous of degree i and j respectively in the jets of the metric and the connection, $\sigma_{a+b-k}(AB)$ is homogeneous of degree $i + j + |\alpha| = k$. \square

Following [BGV], we call generalised Laplacian on E a second order differential operator acting on $C^\infty(M, E)$ with leading symbol $|\xi|^2$. Since generalised Laplacians are expected to be geometric (see the examples in the first section), we assume generalised Laplacians are geometric without further specification. Note that a generalised Laplacian is admissible (see the Appendix). The following result provides a way to build families of geometric operators.

Proposition 1 Let $Q \in C\ell(M, E)$ be a generalised Laplacian with spectral cut² θ . Then for any geometric operator A in $C\ell(M, E)$, the family $A(z) := A Q_\theta^z$ is a family of geometric operators.

²See the Appendix.

Proof: By Lemma 1, it is sufficient to prove the result for $A = I$. For convenience, we drop the explicit mention of the spectral cut.

Since

$$Q^z = \frac{1}{2i\pi} \int_{\Gamma} \lambda^z (Q - \lambda)^{-1} d\lambda,$$

where Γ is a contour described in the Appendix (see formula (56)), we need to investigate the resolvent $R(Q, \lambda) = (Q - \lambda)^{-1}$, the homogeneous components $\sigma_{2-j}(R(Q, \lambda))$ of the symbol of which are defined inductively on j by

$$\begin{aligned} \sigma_{-2}(R(Q, \lambda)) &= (\sigma_2(Q))^{-1}, \\ \sigma_{-2-j}(R(Q, \lambda)) &= -\sigma_{-2}(R(Q, \lambda)) \sum_{k+l+|\alpha|=j, l < j} \frac{(-i)^{|\alpha|}}{\alpha!} D_{\xi}^{\alpha} \sigma_{2-k}(Q) D_x^{\alpha} \sigma_{-2-l}(R(Q, \lambda)) \end{aligned} \quad (6)$$

Using (16), one shows by induction on j that $\sigma_{-2-j}(R(Q, \lambda))$ is a finite sum of expressions of the type

$$(-i)^{|\alpha|} (|\xi|^2 - \lambda)^{-1-k} D_{\xi}^{\alpha_1} D_x^{\beta_1} \sigma_{2-l_1}(Q) \cdots D_{\xi}^{\alpha_k} D_x^{\beta_k} \sigma_{2-l_k}(Q), \quad |l| + |\alpha| = j, \quad |\alpha| = |\beta|.$$

Inserting this in

$$\sigma_{2z-j}(Q^z)(x, \xi) = -\frac{1}{2i\pi} \int_{\Gamma} \lambda^z \sigma_{-2-j}(R(Q, \lambda))(x, \xi) d\lambda, \quad (7)$$

and applying repeated integrations by parts to compute the Cauchy integrals:

$$\frac{1}{2i\pi} \int_{\Gamma} \lambda^z (|\xi|^2 - \lambda)^{-k-1} d\lambda = (-1)^k \frac{z(z+1) \cdots (z+(k-1))}{k!} |\xi|^{2(z-k)},$$

combination of symbols of the type

$$|\xi|^{q(z-k)} D_{\xi}^{\alpha_1} D_x^{\beta_1} \sigma_{q-l_1}(Q)(x, \xi) \cdots D_{\xi}^{\alpha_k} D_x^{\beta_k} \sigma_{q-l_k}(Q)(x, \xi), \quad |l| + |\alpha| = j, \quad |\alpha| = |\beta|. \quad (8)$$

Since $\sigma_{2-l}(Q)$ is homogeneous of order l in the jets of the metric and the connection, it follows that for any complex number z , the symbol $\sigma_{2z-j}(Q^z)$ is homogeneous of order j as a linear combination of products of homogeneous expressions of order j_i in the jets of the metric and the connection such that $j_1 + \cdots + j_k = j$. \square

The notion of geometric operator extends to logarithms of admissible operators defined in the Appendix.

Definition 2 We say that the logarithm $\log_{\theta} A$ (see formula (58) in the Appendix) of an admissible operator $A \in Cl(M, E)$ of order a with spectral cut θ is **geometric** if in any local trivialisation, the homogeneous components $\sigma_{-j,0}(\log_{\theta} A)$ are homogeneous of order j in the jets of the metric and of the connection.

Remark 2 This can be generalised to any log-polyhomogeneous operator A of order a and logarithmic degree k by requiring that all the coefficients $\sigma_{a-j,l}(A)$, $l \in \{0, \dots, k\}$ in the logarithmic expansion of the symbol be homogeneous of order $a-j$ in the jets of the metric and of the connection.

Corollary 1 The logarithm of a generalised Laplacian is a geometric operator.

Proof: Again, we drop the explicit mention of the spectral cut. Differentiating (8) w.r.t. z at zero shows that $\sigma_{-j,0}(\log Q)(x, \xi)$ is a linear combination of symbols of the type

$$|\xi|^{-2k} D_\xi^{\alpha_1} D_x^{\beta_1} \sigma_{2-l_1}(Q)(x, \xi) \cdots D_\xi^{\alpha_k} D_x^{\beta_k} \sigma_{2-l_k}(Q)(x, \xi), \quad |l| + |\alpha| = j, \quad |\alpha| = |\beta|.$$

Hence the symbol $\sigma_{-j,0}(\log Q)$ is homogeneous of order j as a linear combination of products of homogeneous expressions of order j_i in the jets of the metric and the connection such that $j_1 + \cdots + j_k = j$. \square

Remark 3 *This is a particular instance of a more general result, namely that the derivative $A'(0)$ at zero of a holomorphic germ $A(z) \in Cl(M, E)$ of geometric operators around zero, is also geometric.*

Adopting Gilkey's notations (see [G] par. 2.4) let us denote by $\mathcal{P}_{n,k,p}^{g,\nabla^W}$ (which we write $\mathcal{P}_{n,k,p}^g$ if $E = S$) the linear space consisting of p -form valued invariant³ polynomials that are homogeneous of order k in the jets of the metric⁴ and of the connection ∇^W .

Example 3 *The scalar curvature r_M belongs to $\mathcal{P}_{n,2,0}^g$ since it reads $r_M = 2 \sum_{i,j} (\partial_{i,j}^2 g_{ij} - \partial_{i,i}^2 g_{jj})$ in Riemann normal coordinate system.*

Theorem 1 *The logarithmic residue density of a generalised Laplacian Q on E*

$$R_n(x, Q) := \text{res}_x(\log_\theta Q) dx := \frac{1}{(2\pi)^n} \left[\int_{S_x^* M} \text{tr}(\sigma_{-n,0}(\log_\theta Q)(x, \xi)) d\xi \right] dx \quad (9)$$

is an invariant polynomial in $\mathcal{P}_{n,n,n}^{g,\nabla^W}$.

It is generated by Pontrjagin forms of the tangent bundle and Chern forms on the auxillary bundle.

Proof: By Proposition 1 the logarithm (we drop the spectral cut) $\log Q$ is geometric, so that $\sigma_{-n}(\log Q)(x, \xi)$ is homogeneous of degree n in the jets of the metric and the connection. Integrating this expression in ξ on the unit cosphere shows that the residue density lies in $\mathcal{P}_{n,n,n}^{g,\nabla}$.

Since $\mathcal{P}_{n,n,n}^{g,\nabla^W}$ is a polynomial in the 2-jets of the metric and the one jets of the auxillary connection, it is generated by Pontrjagin forms of the tangent bundle (see Theorem 2.6.2 in [G]) and Chern forms on the auxillary bundle. \square

Remark 4 *The logarithmic residue density is clearly additive on direct sums $E_1 \oplus E_2$ of vector bundles over a closed manifold M*

$$R_n(x, Q_1 \oplus Q_2) = R_n(x, Q_1) + R_n(x, Q_2)$$

but there is a priori no reason why it should be multiplicative on tensor products $E_1 \otimes E_2 \rightarrow M_1 \times M_2$ of vector bundles E_i over closed manifolds M_i .

³By invariant we mean that they agree in any coordinate system around x_0 which is normalised w.r. to the point x_0 , i.e. such that $g_{ij}(x_0) = \delta_{i-j}$ and $\partial_k g_{ij}(x_0) = 0$.

⁴The order in the jets of the metric is defined by $\text{ord}(\partial_x^\alpha g_{ij}) = |\alpha|$.

2 The logarithmic residue density via the Campbell-Hausdorff formula

The Campbell-Hausdorff formula provides a first approach to compute a local logarithmic residue density. By the results of Okikiolu [O1], for two admissible classical pseudodifferential operators with scalar leading symbols A and B in $C\ell(M, E)$ and under suitable technical assumptions on their spectrum to ensure that their logarithms are well-defined, we have

$$\log(AB) \sim \log A + \log B + \sum_{k=2}^{\infty} C^{(k)}(\log A, \log B), \quad (10)$$

where $C^{(k)}(\log A, \log B)$ are Lie monomials given by:

$$C^{(k)}(P, Q) := \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(j+1)} \sum \frac{(\text{Ad}_P)^{\alpha_1} (\text{Ad}_Q)^{\beta_1} \cdots (\text{Ad}_P)^{\alpha_j} (\text{Ad}_Q)^{\beta_j}}{(1 + \sum_{l=1}^j \beta_l) \alpha_1! \cdots \alpha_j! \beta_1! \cdots \beta_j!} (Q), \quad (11)$$

which vanish if $\beta_j > 1$ or if $\beta_j = 0$ and $\alpha_j > 1$ and with the inner sum running over j -tuples of pairs (α_i, β_i) such that $\alpha_i + \beta_i > 0$ and $\sum_{i=1}^j \alpha_i + \beta_i = k$. Here $\text{Ad}_P(Q) = [P, Q]$ and the symbol \sim means that for any integer n the difference

$$F_n(A, B) := \log(AB) - \log A - \log B - \sum_{k=2}^{n+1} C^{(k)}(\log A, \log B) \quad (12)$$

is of order smaller than $-n$. The fact that the leading symbols are scalar ensures that the order of $C^{(k)}(\log A, \log B)$ decreases as k increases and hence a good control on the asymptotics as a result of the fact that the adjoint operations $\text{ad}_{\log A}$ and $\text{ad}_{\log B}$ decrease the order by one unit.

Proposition 2 [O1] *Let $A, B \in C\ell(M, E)$ be invertible elliptic operators with scalar leading symbols such that A, B and their product AB have well defined logarithms, then $F_n(A, B)$ defined as in (12) is trace-class so that its Wodzicki residue vanishes. Both its trace and its residue vanish (see also [Sc1]),*

$$\text{res}(\log(AB) - \log A - \log B) = 0. \quad (13)$$

The proof in [O1] is based on a similar expansion on the level of symbols which we now describe for future use. We consider the algebra $\mathcal{FS}(U)$ of formal symbols on an open subset U of \mathbb{R}^n equipped with the symbol product \star

$$\sigma_1 \star \sigma_2(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_{\xi}^{\alpha} \sigma_1(x, \xi) \partial_x^{\alpha} \sigma_2(x, \xi).$$

Let $\{\sigma, \tau\}_{\star} := \sigma \star \tau - \tau \star \sigma$ denote the associated star bracket.

Example 4 *It σ is polynomial, this formal power series of symbols with decreasing order becomes a finite sum, as in the following example of interest to us:*

$$\{|\xi|^2, \tau\}_{\star} = (L_x + \Delta_x) \tau$$

where we have set

$$L_x := -2i \sum_{a=1}^n \xi_a \partial_{x_a} \quad \text{and} \quad \Delta_x := - \sum_{a=1}^n \partial_{x_a}^2. \quad (14)$$

We define ad_σ^{*k} by induction on k setting $\text{ad}_\sigma^{*0}(\tau) = \tau$ and $\text{ad}_\sigma^{*(k+1)}(\tau) := \{\sigma, \text{ad}_\sigma^{*k}(\tau)\}$.

Example 5 $\text{ad}_{|\xi|^2}^{*k}(\tau) = (L_x + \Delta_x)^k \tau$ is a symbol of order $\text{ord}(\tau) + k$.

Here is another example of interest to us.

Example 6

$$\begin{aligned} \{\log |\xi|^2, \tau\}_\star &= \sum_{|\alpha|=0}^{\infty} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \log |\xi|^2 \partial_x^\alpha \tau(x, \xi) \\ &= -2i \sum_{j=1}^n \frac{\xi_j}{|\xi|^2} \partial_{x_j} \tau(x, \xi) - \sum_{i=1}^n \frac{1}{|\xi|^2} \partial_{x_i}^2 \tau(x, \xi) + 2 \sum_{i,j=1}^n \frac{\xi_i \xi_j}{|\xi|^4} \partial_{x_i x_j}^2 \tau(x, \xi) + \dots \end{aligned} \quad (15)$$

We now specialise to the algebra $\mathcal{FS}_{\text{cl}}(U)$ of polyhomogeneous formal symbols. The resolvent of a polyhomogeneous formal symbol σ of order a

$$r_\star(\sigma, \lambda) = (\lambda - \sigma)^{\star-1}, \quad (16)$$

solution of $(\lambda - \sigma) \star r = 1$ has homogeneous components $\sigma_{a-j}(r_\star(\sigma, \lambda))$ of degree $a - j$ in $(\xi, \lambda^{\frac{1}{a}})$ defined inductively on j by

$$\begin{aligned} \sigma_{-a}(r_\star(\sigma, \lambda)) &= (\sigma_a - \lambda)^{-1}, \\ \sigma_{-a-j}(r_\star(\sigma, \lambda)) &= -\sigma_{-a}(r_\star(\sigma, \lambda)) \sum_{k+l+|\alpha|=j, l < j} \frac{(-i)^{|\alpha|}}{\alpha!} D_\xi^\alpha \sigma_{a-k}(\sigma) D_x^\alpha \sigma_{-a-l}(r_\star(\sigma, \lambda)). \end{aligned} \quad (17)$$

Definition 3 We call a formal symbol σ in $\mathcal{FS}_{\text{cl}}(U)$ admissible with spectral cut θ if for every $(x, \xi) \in T^*U - \{0\}$ the leading symbol matrix $\sigma^L(x, \xi)$ has no eigenvalue in a conical neighborhood of the ray $L_\theta = \{re^{i\theta}, r \geq 0\}$. In particular, the symbol is elliptic.

The logarithm of an admissible formal polyhomogeneous symbol σ is defined by (see e.g. [O1]):

$$\log_\star(\sigma) := \frac{i}{2\pi} \left(\partial_z \int_\Gamma \lambda^z (\lambda - \sigma)^{\star-1} d\lambda \right)_{|z=0},$$

for a contour Γ , which encloses the eigenvalues of the leading symbol of σ . The Campbell-Hausdorff formula for admissible formal polyhomogeneous symbols σ and τ with scalar leading symbols reads (see Lemma 2.7 in [O1]):

$$\log_\star(\sigma \star \tau) \sim \log_\star \sigma + \log_\star \tau + \sum_{k=2}^{\infty} C_\star^{(k)}(\log_\star \sigma, \log_\star \tau), \quad (18)$$

where $C_\star^{(k)}(\log_\star \sigma, \log_\star \tau)$ are Lie monomials defined as in (11) replacing $\text{ad}_P(Q)$ by

$$\text{ad}_P^*(q) := \{p, q\}_\star := p \star q - q \star p.$$

The beginning of the expansion in equation (10) reads:

$$\begin{aligned}
\log_\star(\sigma \star \tau) &\sim \log_\star \sigma + \log_\star \tau + \frac{1}{2} \{\log_\star \sigma, \log_\star \tau\}_\star \\
&+ \frac{1}{12} \{\log_\star \sigma, \{\log_\star \sigma, \log_\star \tau\}_\star\}_\star - \frac{1}{12} \{\log_\star \tau, \{\log_\star \sigma, \log_\star \tau\}_\star\}_\star \\
&- \frac{1}{24} \{\log_\star \tau, \{\log_\star \sigma, \{\log_\star \sigma, \log_\star \tau\}_\star\}_\star\}_\star \cdots
\end{aligned} \tag{19}$$

Remark 5 If τ is classical, then $C_\star^{(k)}(\log |\xi|^2, \tau)$ is classical since the bracket $\{\log |\xi|^2, \sigma\}_\star$ with a classical symbol σ is classical.

Remark 6 If τ has negative order, then the order α_k of $C_\star^{(k)}(\log |\xi|^2, \tau)$ is negative and decreases with k . Indeed, α_{k+1} corresponds either to the order of $\left\{ \log |\xi|^2, C_\star^{(k)}(\log |\xi|^2, \tau) \right\}_\star$, which by (15) is $\alpha_k - 1$ or to the order of $\left\{ \tau, C_\star^{(k)}(\log |\xi|^2, \tau) \right\}_\star$ which is $\text{ord}(\tau) + \alpha_k$ and hence smaller than α_k .

Theorem 2 The logarithmic residue density (9) of a generalised Laplacian Q on E is a finite sum of residue densities of classical symbols:

$$\begin{aligned}
\text{res}_x(\log Q) &= \text{res}_x(\log_\star(|\xi|^{-2} \star \sigma(Q)(x, \xi))) \\
&+ \sum_{j=1}^n \frac{(-1)^j}{(j+1)!} \sum_{k=2}^n \text{res}_x \left(C_\star^{(k)} \left(\log |\xi|^2, (|\xi|^{-2} \star \sigma_{<2}(Q)(x, \xi))^{*(j+1)} \right) \right), \\
&= \sum_{j=1}^n \frac{(-1)^j}{(j+1)!} \text{res}_x \left((|\xi|^{-2} \star \sigma_{<2}(Q)(x, \xi))^{*(j+1)} \right) \\
&+ \sum_{j=1}^n \frac{(-1)^j}{(j+1)!} \sum_{k=2}^n \text{res}_x \left(C_\star^{(k)} \left(\log |\xi|^2, (|\xi|^{-2} \star \sigma_{<2}(Q)(x, \xi))^{*(j+1)} \right) \right),
\end{aligned} \tag{20}$$

where we have set $\sigma(Q)(x, \xi) = |\xi|^2 + \sigma_{<2}(Q)(x, \xi)$.

Proof: We write $\sigma(Q)(x, \xi) = |\xi|^2 \star (1 + |\xi|^{-2} \star \sigma_{<2}(Q)(x, \xi))$. Applying the Campbell-Hausdorff formula (18) to $\sigma = |\xi|^2$ and $\tau = 1 + |\xi|^{-2} \star \sigma_{<2}(Q)$ yields:

$$\begin{aligned}
\sigma(\log Q)(x, \xi) &\sim \log_\star \sigma(Q)(x, \xi) \\
&\sim 2 \log |\xi| + \log_\star(|\xi|^{-2} \star \sigma(Q)(x, \xi)) \\
&+ \sum_{k=2}^{\infty} C_\star^{(k)}(\log |\xi|^2, \log_\star(1 + |\xi|^{-2} \star \sigma_{<2}(Q))) \\
&\sim 2 \log |\xi| + \sum_{j=1}^n \frac{(-1)^j}{(j+1)!} (|\xi|^{-2} \star \sigma_{<2}(Q)(x, \xi))^{*(j+1)} \\
&+ \sum_{k=2}^{\infty} C_\star^{(k)}(\log |\xi|^2, \log_\star(1 + |\xi|^{-2} \star \sigma_{<2}(Q))).
\end{aligned}$$

This shows that $\log_\star \sigma(Q) - \log |\xi|^2$ is a classical symbol as a consequence of the fact that the logarithm $\log_\star \tau \sim \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} (|\xi|^{-2} \star \sigma_{<2}(Q)(x, \xi))^{*(j+1)}$, is classical and

hence that the corresponding Lie monomials are also classical by Remark 5. Applying Remark 6 to $\tau = \log_\star (1 + |\xi|^{-2} \star \sigma_{<2}(Q))$, which has negative order, shows that $C_\star^{(k)} (\log |\xi|^2, \log_\star (1 + |\xi|^{-2} \star \sigma_{<2}(Q)))$ has order smaller than $-k$. Since the residue vanishes on symbols of order smaller than $-n$ and $(|\xi|^{-2} \star \sigma_{<2}(Q)(x, \xi))^{*(j+1)}$ has order no larger than $-(j+1)$, implementing the residue yields:

$$\begin{aligned} \text{res}_x(\log Q) &= \text{res}_x(\log_\star (|\xi|^{-2} \star \sigma(Q)(x, \xi))) \\ &+ \sum_{k=2}^n \text{res}_x \left(C_\star^{(k)} (\log |\xi|^2 \star \log_\star (1 + |\xi|^{-2} \star \sigma_{<2}(Q))) \right). \end{aligned}$$

Replacing $\log_\star (|\xi|^{-2} \star \sigma(Q)(x, \xi)) = \log_\star (1 + |\xi|^{-2} \star \sigma_{<2}(Q))$ by its expansion yields the result. \square

3 The logarithmic residue density via a noncommutative Taylor expansion

A noncommutative Taylor type formula provides an alternative way to express logarithmic residue densities. We extend formulae for noncommutative Taylor expansions derived in [P] to formal polyhomogeneous symbols.

Given an analytic function $\phi(z) = \phi_0 + \phi_1 z + \phi_2 z^2 + \dots$ and an admissible symbol σ in $\mathcal{FS}_{\text{cl}}(U)$ we write

$$\Phi_\star(\sigma) = \frac{1}{2i\pi} \int_\Gamma r_\star(\lambda, \sigma) \phi(\lambda) d\lambda. \quad (21)$$

where the resolvent $r_\star(\lambda, \sigma)$ is defined by (16) and where Γ is a contour which encloses the eigenvalues of the leading symbol of σ . Applying this to the higher derivative $\phi^{(k)}$ yields:

$$\Phi_\star^{(k)}(\sigma) = \frac{1}{2i\pi} \int_\Gamma (\lambda - r_\star(\lambda, \sigma)) \phi^{(k)}(\lambda) d\lambda = \frac{k!}{2i\pi} \int_\Gamma (\lambda - \sigma)^{*(-k-1)} \phi(\lambda) d\lambda. \quad (22)$$

If $\sigma = |\xi|^q + \sigma_{<q}$ with $\sigma_{<q}$ of order smaller than q , then the \star -resolvent reads:

$$r_\star(\lambda, |\xi|^q + \sigma_{<q}) = r_\star(\lambda, |\xi|^q) + \sum_{n=1}^{\infty} r_{\star n}(\lambda, |\xi|^q) (\sigma_{<q})^{\otimes n}, \quad (23)$$

where for symbols τ_1, \dots, τ_n in $\mathcal{FS}_{\text{cl}}(U)$ we have set

$$\begin{aligned} &r_{\star n}(\lambda, |\xi|^q)(\tau_1 \otimes \dots \otimes \tau_n) \\ &= \sum_{|k|=0}^{\infty} \frac{(k_1 + \dots + k_n + n - 1)!}{k! (k_1 + 1)(k_1 + k_2 + 1) \dots (k_1 + \dots + k_{n-1} + n - 1)} \\ &\cdot \text{ad}_{|\xi|^q}^{(k_1)}(\tau_1) \star \text{ad}_{|\xi|^q}^{(k_2)}(\tau_2) \star \dots \star \text{ad}_{|\xi|^q}^{(k_n)}(\tau_n) (\lambda - |\xi|^q)^{-|k|-n-1}, \end{aligned} \quad (24)$$

and $|k| = k_1 + \dots + k_n$ and $k! = k_1! \dots k_n!$.

Second quantised functionals are defined on tensor products of symbols in terms of Cauchy integrals in analogy to ordinary functionals on symbols (see (21)), but by means of quantised resolvents $r_{\star n}$ instead of the ordinary resolvent r_\star .

To an analytic function $\phi(z)$ and to an admissible symbol σ we assign a map called *second quantisation of $\Phi(x)$* defined on $(\mathcal{F}S_{\text{cl}}(U))^{\otimes n}$ by

$$\begin{aligned}\Phi_{\star n}(\sigma) : \quad (\mathcal{F}S_{\text{cl}}(U))^{\otimes n} &\rightarrow \mathcal{F}S_{\text{cl}}(U) \\ \tau_1 \otimes \cdots \otimes \tau_n &\mapsto \frac{1}{2i\pi} \int_{\Gamma} r_{\star n}(\lambda, \sigma)(\tau_1 \otimes \cdots \otimes \tau_n) \phi(\lambda) d\lambda.\end{aligned}$$

One easily derives the following noncommutative Taylor type formula from (24):

$$\Phi_{\star n}(\sigma)(\tau_1 \otimes \cdots \otimes \tau_n) = \sum_{|k|=0}^{\infty} \frac{\text{ad}_{\sigma}^{\star k_1}(\tau_1) \star \cdots \star \text{ad}_{\sigma}^{\star k_n}(\tau_n)}{k! (k_1+1)(k_1+k_2+2) \cdots (k_1+\cdots+k_n+n)} \Phi^{(|k|+n)}(\sigma). \quad (25)$$

Applying (23) to $\sigma = |\xi|^2 + \sigma_{<2}$ where $\sigma_{<2}$ has order smaller than 2 we have:

$$\Phi_{\star}(\sigma) = \Phi_{\star}(|\xi|^q) + \sum_{p=1}^{\infty} \Phi_{\star p}(|\xi|^q)(\sigma_{<2}^{\otimes p}) \quad (26)$$

Applying this to $\phi = \log$ yields

$$\begin{aligned}& \log_{\star}(\sigma) - \log_{\star}(|\xi|^2) \\ &= \sum_{p=1}^{\infty} \sum_{|k|=0}^{\infty} (-1)^{|k|+p-1} (|k|+p-1)! \frac{\text{ad}_{|\xi|^2}^{\star k_1}(\sigma_{<2}) \star \text{ad}_{|\xi|^2}^{\star k_2}(\sigma_{<2}) \cdots \star \text{ad}_{|\xi|^2}^{\star k_p}(\sigma_{<2})}{k! (k_1+1)(k_1+k_2+2) \cdots (k_1+\cdots+k_p+p)} |\xi|^{-2(|k|+p)}.\end{aligned} \quad (27)$$

Implementing the noncommutative residue finally leads to the following formula for the logarithmic residue density.

Theorem 3 *The logarithmic residue density of a generalised Laplacian Q on E is a finite sum of residues of classical symbols:*

$$\begin{aligned}& \text{res}_x(\log(Q)) \\ &= \sum_{p=1}^n \sum_{|k|=0}^{n-p} (-1)^{|k|+p-1} (|k|+p-1)! \frac{\text{res}_x((L_x + \Delta_x)^{k_1}(\sigma_{<2}(Q)) \cdots (L_x + \Delta_x)^{k_p}(\sigma_{<2}(Q)) |\xi|^{-2(|k|+p)})}{k! (k_1+1)(k_1+k_2+2) \cdots (k_1+\cdots+k_p+p)},\end{aligned} \quad (28)$$

where $k! := k_1! \cdots k_p!$ and $|k| = k_1 + \cdots + k_p$. Here we have set $\sigma_{<2}(Q)(x, \xi) := \sigma(Q)(x, \xi) - |\xi|^2$ and as before, $L_x := -2i \sum_{a=1}^n \xi_a \partial_{x_a}$ and $\Delta_x := -\sum_{a=1}^n \partial_{x_a}^2$.

Proof: By (27) combined with (5) we have:

$$\begin{aligned}& \sigma(\log(Q))(x, \xi) \\ &\sim \sum_{p=1}^{\infty} \sum_{|k|=0}^{\infty} (-1)^{|k|+p-1} (|k|+p-1)! \frac{(L_x + \Delta_x)^{k_1}(\sigma_{<2}(Q)) \cdots (L_x + \Delta_x)^{k_p}(\sigma_{<2}(Q)) |\xi|^{-2(|k|+p)}}{k! (k_1+1)(k_1+k_2+2) \cdots (k_1+\cdots+k_p+p)},\end{aligned}$$

which is a formal power series of symbols σ_k of decreasing order $-(|k|+p)$. Since the noncommutative residue vanishes on symbols of order smaller than $-n$, we have $|k|+p \leq n$ which implies that only terms $p \leq n$ and $|k| \leq n-p$ survive after applying the residue. \square

4 The index as a logarithmic (super-) residue

Let us recall results of [PS] and [Sc1] (see also [Sc2]). Let $Q \in C\ell(M, E)$ be an admissible (and hence invertible, see Appendix) classical pseudodifferential operator of positive order q .

For any *differential* operator $A \in C\ell(M, E)$, the noncommutative residue density

$$\text{res}_x(A \log Q) dx := -\frac{1}{(2\pi)^n} \left(\int_{|\xi|=1} \text{tr}(\sigma_{-n}(A \log Q)(x, \xi)) d_S \xi \right) dx, \quad (29)$$

is a globally defined n -form on M (see [O2] for the case $A = I$, [PS] for the general case), which integrates over M to the noncommutative residue:

$$\text{res}(A \log Q) := -\frac{1}{(2\pi)^n} \int_M \left(\int_{|\xi|=1} \text{tr}(\sigma_{-n}(A \log Q)(x, \xi)) d_S \xi \right) dx. \quad (30)$$

It furthermore relates to the Q -weighted trace $\text{Tr}^Q(A)$ of A by (see [Sc1] when $A = I$ and [PS] for the general case)

$$\text{Tr}^Q(A) := \text{fp}_{z=0} \text{TR}(A Q^{-z}) = -\frac{1}{q} \text{res}(A \log_\theta Q), \quad (31)$$

where $\text{fp}_{z=0}$ stands for the finite part at $z = 0$. Here, $d_S \xi$ is the volume form on the unit sphere induced by the canonical measure on \mathbb{R}^n , where σ_{-n} stands for the positively homogeneous component of degree $-n$ of a logpolyhomogeneous symbol σ .

Remark 7 *Once checks that $\text{res}(A \log(Q + R)) = \text{res}(A \log Q)$ for any smoothing operator R .*

Example 7 *Setting $A = I$ in the above corollary yields*

$$\zeta_Q(0) = -\frac{1}{q} \text{res}(\log Q), \quad (32)$$

where $\zeta_Q(z)$ is the zeta function associated to Q . This corresponds to the logarithm $\text{res}(\log Q) = \log \det_{\text{res}}(Q)$ of Scott's residue determinant [Sc1].

Let $E = E^+ \oplus E^-$ be any \mathbb{Z}_2 graded vector bundle over M and let $D^+ : C\ell(M, E_+) \rightarrow C\ell(M, E_-)$ be an elliptic operator in $C\ell(M, E_+^* \otimes E_-)$. Its (formal) adjoint $D^- := (D^+)^* : C\ell(M, E^-) \rightarrow C\ell(M, E^+)$ is an elliptic operator in $C\ell(M, (E^-)^* \otimes E_+)$ and $\Delta = \Delta^+ \oplus \Delta^-$ with $\Delta^+ := D^- D^+$, $\Delta^- := D^+ D^-$ are non-negative (formally) self-adjoint elliptic operators.

The following theorem which combines formulae due to McKean and Singer [MS] and Seeley [Se], expresses the index of D^+ :

$$\text{ind}(D^+) := \dim(\text{Ker}(D^+)) - \dim(\text{Ker}(D^-))$$

in terms of the superweighted trace of the identity. Let π_Δ denote the orthogonal projection onto the kernel of Δ , which is finite dimensional as M is compact.

Theorem 4 *The superresidue*

$$\text{sres}(\log(\Delta)) := -\frac{1}{(2\pi)^n} \left(\int_{|\xi|=1} \text{str}(\sigma_{-n,0}(\log(\Delta))(x, \xi)) d_S \xi \right) dx,$$

is a globally defined n -form and we have

$$\text{ind}(D^+) = \text{sTr}^{\Delta+\pi_\Delta}(I) = -\frac{1}{2 \text{ord}(D)} \text{sres}(\log(\Delta + \pi_\Delta)), \quad (33)$$

where π_Δ is the orthogonal projection onto the kernel of Δ and $\text{ord}(D)$ is the order of D . Here str stand for the super trace on the graded fibres of E .

Remark 8 *In view of Remark 7, one can drop the explicit mention of π_Δ and write $\text{sres}(\log \Delta)$ since the projection π_Δ is smoothing and the residue is invariant under translation by a smoothing operator.*

Proof: We first observe a property of the spectrum of Δ :

$$\text{Spec}(\Delta^+) - \{0\} = \text{Spec}(\Delta^-) - \{0\}.$$

Indeed,

$$\Delta^+ u_+ = \lambda^+ u_+ \Rightarrow \Delta_-(D^+ u_+) = \lambda^+ D^+ u_+ \quad \forall u_+ \in C^\infty(M, E_+)$$

so that an eigenvalue λ^+ of Δ^+ with eigenvector u_+ is an eigenvalue of Δ^- with eigenvector $D^+ u_+$ provided the latter does not vanish. The converse holds similarly. Let us denote by $\{\lambda_n^+, n \in \mathbb{N}\}$ the discrete set of eigenvalues of Δ^+ and by $\{\lambda_n^-, n \in \mathbb{N}\}$ the discrete set of eigenvalues of Δ^- . For any complex number z :

$$\begin{aligned} \text{sTr}((\Delta + \pi_\Delta)^{-z}) &= \sum_{n \in \mathbb{N}} (\lambda_n^+ + \delta_{\lambda_n^+})^{-z} - \sum_{n \in \mathbb{N}} (\lambda_n^- + \delta_{\lambda_n^-})^{-z} \\ &= \sum_{\lambda_n^+ \neq 0} (\lambda_n^+)^{-z} - \sum_{\lambda_n^- \neq 0} (\lambda_n^-)^{-z} + \dim \text{Ker} \Delta^+ - \dim \text{Ker} \Delta^- \\ &= \text{ind}(D^+). \end{aligned}$$

Taking the finite part at $z = 0$ therefore yields:

$$\text{ind}(D^+) = \text{sTr}^{\Delta+\pi_\Delta}(I) = -\frac{1}{2} \text{sres}(\log(\Delta)).$$

□

Example 8 *With the notations introduced at the beginning of the paper, for a Dirac operator $D_W^+ : C^\infty(M, S^+ \otimes W) \rightarrow C^\infty(M, S^- \otimes W)$ on the \mathbb{Z}_2 -graded spinor bundle $S = S^+ \oplus S^-$ over an even dimension spin manifold M we have*

$$\text{ind}(D_W^+) = -\frac{1}{2} \text{sres}(\log(D_W^2)) = -\frac{1}{2} \int_M \text{sres}_x(\log(D_W^2)) dx. \quad (34)$$

The remaining part of the paper deals with the computation of the logarithmic density of the square D^2 of the Dirac operator D acting on spinors.

5 A formula for the index via the Lichnerowicz formula

We first recall the Lichnerowicz formula (see e.g. Theorem 3.52 of [BGV]) or equivalently the general Bochner identity (see Theorem 8.2 of [LM]), which relates the square D_W^2 of the twisted Dirac operator D_W with the Laplace-Beltrami operator

$$\Delta^E = -\text{tr} \left(\nabla^{T^*M \otimes E} \nabla^E \right) = - \sum_{i=1}^n \left(\nabla^{T^*M \otimes E} \nabla^E \right)_{e_i, e_i} = - \sum_{i=1}^n \left(\nabla_{e_i}^E \nabla_{e_i}^E - \nabla_{\nabla_{e_i}^E e_i}^E \right) \quad (35)$$

associated with the superconnection ∇^E on E , where $\nabla^{T^*M \otimes E}$ is the connection induced on the tensor product bundle $T^*M \otimes E$ by the Levi-Civita connection on M and the connection ∇^E on E . Here $\{e_i, i = 1, \dots, n\}$ is a local orthonormal tangent frame.

Proposition 3

$$D_W^2 = \Delta^E + R^E = \Delta^E + R^W + \frac{r_M}{4}, \quad (36)$$

where r_M stands for the scalar curvature on M and

$$R^E := \sum_{i < j} c(e_i) c(e_j) (\nabla^E)_{e_i, e_j}^2; \quad R^W := \sum_{i < j} c(e_i) c(e_j) (\nabla^W)_{e_i, e_j}^2. \quad (37)$$

In particular, for a flat auxiliary bundle we have:

$$D_W^2 = \Delta_M + \frac{r_M}{4},$$

where Δ_M is the Laplace-Beltrami operator on the Riemannian manifold M .

Proof: We choose a local orthonormal tangent frame $\{e_i, i = 1, \dots, n\}$ at point $x \in M$ such that $(\nabla_{e_i}^E)_x = 0$ for all $i \in \{1, \dots, n\}$. Since $D_W = \sum_{i=1}^n c(e_i) \nabla_{e_i}^E$, at that point x we have:

$$\begin{aligned} D_W^2 &= \sum_{i,j=1}^n c(e_i) \nabla_{e_i}^E c(e_j) \nabla_{e_j}^E \\ &= \sum_{i,j=1}^n c(e_i) c(e_j) \left[(\nabla^E)_{e_i, e_j}^2 + \nabla_{\nabla_{e_i}^E e_j}^E \right] \\ &= - \sum_{i=1}^n (\nabla^E)_{e_i, e_i}^2 + \sum_{i < j} c(e_i) c(e_j) \left[(\nabla^E)_{e_i, e_j}^2 - (\nabla^E)_{e_j, e_i}^2 \right] \\ &= \Delta^E + \sum_{i < j} c(e_i) c(e_j) (\nabla^E)_{e_i, e_j}^2 \\ &= \Delta^E + R^E. \end{aligned}$$

The curvature term $(\nabla^E)^2 \in \Omega^2(M, \text{End}(E))$ decomposes as $(\nabla^E)^2 = (\nabla^S)^2 \otimes 1 + 1 \otimes (\nabla^W)^2$ so that $R^E = \sum_{i < j} c(e_i) c(e_j) (\nabla^S)_{e_i, e_j}^2 + R^W$. A careful computation (see e.g. the proof of Theorem 3.52 in [BGV]) shows that $\sum_{i < j} c(e_i) c(e_j) (\nabla^S)_{e_i, e_j}^2 = \frac{r_M}{4}$. \square

Combining the Lichnerowicz formula with the Campbell-Hausdorff formula yields a formula for the index.

Theorem 5 *In even dimension $n = 2p$,*

$$\begin{aligned} \text{ind}(D_W^+) &= -\frac{1}{2} \text{sres}(\log(D_W^2)) \\ &= -\frac{1}{2} \text{sres}(\log(\Delta^E)) + \sum_{k=1}^{n-1} \frac{(-1)^k}{2k} \text{sres}\left([\Delta^E]^{-1} R^E\right)^k. \end{aligned} \quad (38)$$

Inside the residue we write for short $(\Delta^E)^{-1}$ instead of $(\Delta^E + \pi_\Delta)^{-1}$ since the residue is insensitive to the smoothing operator π_Δ .

Proof: By equation (36)

$$\begin{aligned} D_W^2 + \pi_{D_W^2} &= \Delta^E + \pi_{\Delta^E} + R^E + \pi_{D_W^2} - \pi_{\Delta^E} \\ &= (\Delta^E + \pi_{\Delta^E}) \left(1 + (\Delta^E + \pi_{\Delta^E})^{-1} (R^E + \pi_{D_W^2} - \pi_{\Delta^E})\right), \end{aligned}$$

so that by (13), we get:

$$\begin{aligned} &\text{sres}(\log(D_W^2)) \\ &= \text{sres}(\log(\Delta^E)) + \text{sres}\left(\log\left(1 + (\Delta^E)^{-1} (R^E)\right)\right) \\ &= \text{sres}(\log(\Delta^E)) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \text{sres}\left([\Delta^E]^{-1} (R^E)\right)^k \\ &= \text{sres}(\log(\Delta^E)) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \text{sres}\left([\Delta^E]^{-1} R^E\right)^k. \end{aligned}$$

Here we used the fact that the noncommutative residue vanishes on smoothing operators. Also, for an operator $B \in \mathcal{Cl}(M, E)$ with negative order, we have $\text{sres}(\log(1 + B)) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \text{sres}(B^k)$, which is actually a finite sum since the residue vanishes for operators of order smaller than minus the dimension of the underlying manifold. Since $B = (\Delta^E + \pi_\Delta)^{-1} R^E$ has order -2 , the sum stops at $p = n/2$. \square

6 The Atiyah-Singer index theorem for a twisted Dirac operator on a flat space

We derive the Atiyah-Singer index formula for a twisted Dirac operator on a flat space from (38). We use the notations introduced at the beginning of the paper. Denoting by $\partial_i + A_i$ the components of the connection ∇^E in a given local trivialization of E and local coordinates x_i on M such that the metric Christoffel symbols vanish, according to the Lichnerowicz formula we have

$$D_W^2 = \Delta^E + R^E = \sum_i (\partial_i + A_i)^2 + \sum_{i < j} \gamma_i \gamma_j F_{ij},$$

where we have set $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ to be the 2-form components of the curvature $(\nabla^E)^2$.

Theorem 6 *If the Riemann metric on M is flat, then*

$$\text{ind}(D_W^+) = \int_M \text{tr} \left(e^{i \frac{F}{2\pi}} \right).$$

Proof: As before, $n = 2p$ stands for the dimension of M . By Theorem 5 we have

$$\text{sres} \log(D_W^2) = \text{sres}(\log \Delta^E) + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \text{sres} \left([(\Delta^E)^{-1} R^E]^k \right).$$

The first term on the r.h.s. vanishes. Indeed, at a given point $x \in M$ and for fixed $\xi \in T_x^* M$, the $-n$ -th homogeneous component of the symbol $\sigma(\log \Delta^E)(x, \xi)$ of $\log \Delta^E$ is an endomorphism of the fibre W_x of the auxillary vector bundle W . By (3) the fibrewise supertrace therefore vanishes on the $-n$ -th homogeneous component of the symbol and hence so does the residue density $\text{sres}_x(\log \Delta^E) dx$. Thus $\text{sres}(\log \Delta^E) = 0$. We now investigate the second term on the r.h.s. On the one hand, all the expressions $\text{sres} \log \left([(\Delta^E)^{-1} R^E]^k \right)$ inside the sum vanish for $k > p$, for the

operators $[(\Delta^E)^{-1} R^E]^k$ being of order smaller than $-n$, their residues vanish.

On the other hand, the expressions inside the sum also vanish for $k < p$. Indeed, at a point $x \in M$ and for fixed $\xi \in T_x^* M$, the symbols in the variables (x, ξ) inside the residues are of the form $M \gamma_1 \gamma_2 \cdots \gamma_k$ for some matrix $M \in \text{End}(W_x)$ and sets $\{i_1, \dots, i_k\}$ strictly smaller than $\{1, \dots, n\}$. Their supertrace which arise inside the superresidue, therefore vanishes by (3).

The remaining $k = p$ term in the sum, which corresponds to the residue of an operator of order $-n$, only involves the leading symbol $\sigma_L(\Delta^E) = |\xi|^2$ of Δ^E . Thus we obtain

$$\begin{aligned} \text{sres} \log(D_W^2) &= -\frac{(-1)^p}{p} \text{sres} \left([(\Delta^E + \pi_{\Delta^E})^{-1} R^E]^p \right) \\ &= -\frac{(-1)^p}{p} \text{sres} \left(|\xi|^{-n} (\text{tr}(R^E))^p \right) \\ &= -\frac{(-1)^p}{p} \int_M \text{sres}_x \left(|\xi|^{-n} \left(\sum_{i < j} \gamma_i \gamma_j F_{ij} \right)^p \right) dx \quad \text{by (37)} \\ &= -\frac{(-1)^p 2^p \text{Vol}(S^{n-1})}{(2\pi)^n p} \int_M \text{str} \left(\sum_{i,j} \sigma_{ij} F_{ij} \right)^p dx \quad \text{by (4) and (1)} \\ &= -2 \frac{i^p}{(4\pi)^p p!} \sum_{\tau \in \Sigma_n} (-1)^{|\tau|} \int_M F_{\tau(1)\tau(2)} \cdots F_{\tau(n-1)\tau(n)} dx \quad \text{by (5) and (39)} \\ &= -2 \frac{i^p}{(2\pi)^p p!} \int_M \text{tr}(F^{\wedge p}) dx \\ &= -2 \int_M \text{tr} \left(e^{i \frac{F}{2\pi}} \right). \end{aligned}$$

Here, we have used

$$\text{vol}(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \frac{2\pi^p}{(p-1)!}. \quad (39)$$

□

7 The curvature tensor in normal coordinates

We recall a few properties of the curvature in a normal local coordinate system, i.e., a coordinate system defined by the exponential map at a point, so that rays emanating from the origin in the tangent space at a point are mapped to geodesics on the manifold emanating from this point. Let us recall that in Riemannian normal coordinates (see e.g. Proposition 1.28 in [BGV]),

$$g_{ij} = \partial_{ij} - \frac{1}{3}R_{ikjl}x^l x^k + \sum_{|\alpha| \geq 3} \partial_\alpha g_{ij} \frac{x^\alpha}{\alpha!} \quad (40)$$

Lemma 2 *We have*

$$(R_{iajk} + R_{ikja})\sigma_{kj} = \frac{3}{2}R_{iajk}\sigma_{kj},$$

where σ_{ij} was defined in (4).

Proof: Using the first Bianchi identity

$$R_{[ijk]l} = 0$$

we write $R_{ijka} = -R_{kija} - R_{jkia}$, which combined with the antisymmetry of σ_{ij} in i and j and the (anti)symmetry properties of the curvature tensor $R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$ yields:

$$\begin{aligned} (R_{iajk} + R_{ikja})\sigma_{kj} &= R_{iajk}\sigma_{kj} + R_{ijk a}\sigma_{jk} \\ &= (R_{iajk} + R_{kija} + R_{jkia})\sigma_{kj} \\ &= (2R_{iajk} - R_{ikja})\sigma_{kj}. \end{aligned}$$

Consequently,

$$\begin{aligned} (R_{iajk} + R_{ikja})\sigma_{kj} &= \frac{1}{2}[(R_{iajk} + R_{ikja})\sigma_{kj} + (2R_{iajk} - R_{ikja})\sigma_{kj}] \\ &= \frac{3}{2}R_{iajk}\sigma_{kj} \end{aligned}$$

□

Proposition 4 *At the center of a normal coordinate system, we have*

$$\partial_a \Gamma_{ij}^k \sigma_{kj} = \frac{1}{2}R_{jkia}\sigma_{kj} \quad (41)$$

so that $\partial_i \Gamma_{ij}^k \sigma_{kj} = 0$.

Proof: By (40) the Christoffel symbols $\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij})$ vanish at the center of the normal coordinate system, where we have:

$$\partial_a \Gamma_{ij}^k = \frac{1}{3}(R_{iajk} + R_{j aik}).$$

Indeed, differentiating (40) twice yields

$$\begin{aligned} \partial_a \Gamma_{ij}^k &= \frac{1}{2}\delta^{kl}(\partial_a \partial_j g_{il} + \partial_a \partial_i g_{jl} - \partial_a \partial_l g_{ij}) \\ &= -\frac{1}{6}(R_{iakj} + R_{ijka} + R_{jaki} + R_{jika} - R_{iajk} - R_{ikja}) \\ &= \frac{1}{3}(R_{iajk} + R_{j aik}). \end{aligned}$$

It follows from Lemma 2 that $\partial_a \Gamma_{ij}^k \sigma_{kj} = \frac{1}{2} R_{iajk} \sigma_{kj}$. and hence in particular that $\partial_i \Gamma_{ij}^k \sigma_{kj} = \frac{1}{2} R_{iajk} \sigma_{kj} = 0$. at the center of the normal coordinate system. \square
The following result is useful to compute the index.

Proposition 5 *In four dimensions we have:*

$$\text{sres}_x (|\xi|^{-4} \partial_{x_a} \Gamma_{ij}^k \partial_{x_a} \Gamma_{im}^n \sigma_{kj} \sigma_{nm}) dx = \frac{1}{32 \pi^2} \text{tr}(R \wedge R), \quad (42)$$

and

$$\text{sres}_x \left(\frac{\xi_a \xi_b}{|\xi|^6} \partial_{x_a} \Gamma_{ij}^k \partial_{x_b} \Gamma_{im}^n \right) dx = \frac{1}{4 \times 32 \pi^2} \text{tr}(R \wedge R). \quad (43)$$

Proof: The result in four dimensions is a consequence of the following formula in $n = 2p = 4q$ dimensions. At the center of a normal coordinate system we show that:

$$\begin{aligned} & \text{sres}_x \left(|\xi|^{-2p} \partial_{x_{a_1}} \Gamma_{i_1 j_1}^{k_1} \partial_{x_{a_1}} \Gamma_{i_1 m_1}^{n_1} \cdots \partial_{x_{a_p}} \Gamma_{i_q j_q}^{k_q} \partial_{x_{a_q}} \Gamma_{i_q m_q}^{n_q} \sigma_{k_1 j_1} \sigma_{n_1 m_1} \cdots \sigma_{k_q j_q} \sigma_{n_q m_q} \right) dx \\ &= \frac{1}{\Gamma(p) 2^{3p-1} \pi^p} (\text{tr}(R \wedge R))^q. \end{aligned} \quad (44)$$

The proof follows from combining (41) with (29) and the formula for the volume of the unit sphere S^{n-1} in n dimensions given by (39):

$$\begin{aligned} & \text{sres}_x \left(|\xi|^{-2p} \partial_{x_{a_1}} \Gamma_{i_1 j_1}^{k_1} \partial_{x_{a_1}} \Gamma_{i_1 m_1}^{n_1} \cdots \partial_{x_{a_q}} \Gamma_{i_q j_q}^{k_q} \partial_{x_{a_q}} \Gamma_{i_q m_q}^{n_q} \sigma_{k_1 j_1} \sigma_{n_1 m_1} \cdots \sigma_{k_q j_q} \sigma_{n_q m_q} \right) \\ &= \frac{1}{2^{2q}} \text{sres}_x (|\xi|^{-2p} R_{i_1 a_1 j_1 k_1} R_{i_1 a_1 m_1 n_1} \cdots R_{i_q a_q j_q k_q} R_{i_q a_q m_q n_q} \sigma_{k_1 j_1} \sigma_{n_1 m_1} \cdots \sigma_{k_q j_q} \sigma_{n_q m_q}) \quad \text{by (41)} \\ &= \frac{(-i)^p}{4^p} \sum_{\tau \in \Sigma_n} (-1)^{|\tau|} \text{res}_x (|\xi|^{-2p} R_{i_1 a_1 \tau(1) \tau(2)} R_{i_1 a_1 \tau(3) \tau(4)} \cdots R_{i_q a_q \tau(n-3) \tau(n-2)} R_{i_q a_q \tau(n-1) \tau(n)}) \\ &\quad \text{by (3) and (5)} \\ &= \frac{1}{\Gamma(p) 2^{2n-1} \pi^p} \sum_{\tau \in \Sigma_n} (-1)^{|\tau|} R_{i_1 a_1 \tau(1) \tau(2)} R_{i_1 a_1 \tau(3) \tau(4)} \cdots R_{i_q a_q \tau(n-3) \tau(n-2)} R_{i_q a_q \tau(n-1) \tau(n)}. \end{aligned}$$

On the other hand we have

$$\frac{1}{2^p} \sum_{\tau \in \Sigma_n} (-1)^{|\tau|} R_{i_1 a_1 \tau(1) \tau(2)} R_{i_1 a_1 \tau(3) \tau(4)} \cdots R_{i_q a_q \tau(n-3) \tau(n-2)} R_{i_q a_q \tau(n-1) \tau(n)} dx = (\text{tr}(R \wedge R))^q,$$

which yields

$$\begin{aligned} & \text{sres}_x \left(|\xi|^{-2p} \partial_{x_{a_1}} \Gamma_{i_1 j_1}^{k_1} \partial_{x_{a_1}} \Gamma_{i_1 m_1}^{n_1} \cdots \partial_{x_{a_q}} \Gamma_{i_q j_q}^{k_q} \partial_{x_{a_q}} \Gamma_{i_q m_q}^{n_q} \sigma_{k_1 j_1} \sigma_{n_1 m_1} \cdots \sigma_{k_q j_q} \sigma_{n_q m_q} \right) dx \\ &= \frac{1}{\Gamma(p) 2^{3p-1} \pi^p} (\text{tr}(R \wedge R))^q. \end{aligned}$$

This proves the first part of the statemnt. We prove the second part similarly. We first observe that using the symmetries of the sphere we have

$$\int_{|\xi|=1} \frac{\xi_i \xi_j}{|\xi|^{n+2}} d\xi = \delta_{i-j} \int_{|\xi|=1} \frac{\xi_i^2}{|\xi|^{n+2}} d\xi = \frac{1}{n} \delta_{i-j} \int_{|\xi|=1} \frac{\sum_{i=1}^n \xi_i^2}{|\xi|^{n+2}} d\xi = \frac{\delta_{i-j}}{n} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \quad (45)$$

so that in four dimensions we get:

$$\text{sres}_x \left(\frac{\xi_a \xi_b}{|\xi|^6} \partial_{x_a} \Gamma_{ij}^k \partial_{x_b} \Gamma_{im}^n \right) dx = \frac{1}{4 \times 32 \pi^2} \text{tr}(R \wedge R).$$

\square

8 The Atiyah-Singer index formula in dimension 4

The square of a Dirac operator D acting on pure spinors is the prototype of a generalised Laplacian. In local coordinates, its symbol reads:

$$\sigma(D^2) = |\xi|^2 + \Gamma_{ij}^k \sigma_{kj} \xi_i + \partial_i \Gamma_{ij}^k \sigma_{kj} + \Gamma_{ij}^k \Gamma_{im}^n \sigma_{kj} \sigma_{nm} + s = |\xi|^2 + \sigma_{<2}(D^2),$$

where

$$\sigma_{<2}(D^2) := \Gamma_{ij}^k \sigma_{kj} \xi_i + \partial_i \Gamma_{ij}^k \sigma_{kj} + \Gamma_{ij}^k \Gamma_{im}^n \sigma_{kj} \sigma_{nm} + s, \quad (46)$$

and where s stands for the scalar curvature.

We use (34) to compute the index of D^+ :

$$\text{ind}(D^+) = -\frac{1}{2} \int_M \text{sres}_x(\log D^2) dx,$$

in terms of the logarithmic (super) residue density, which we explicitly derive in four dimensions.

Since the residue does not depend on the choice of local coordinate, we choose to derive the residue density in a normal coordinate system. We therefore need to compute:

$$\text{sres}_x(\log D^2) = \text{sres}_x(\log_\star(|\xi|^2 + \sigma_{<2}(D^2))),$$

where \log_\star is the logarithm on symbols. There are at least two methods to compute the logarithm of $|\xi|^2 + \sigma_{<2}(D^2)$, the Campbell-Hausdorff formula (18) and a Taylor type formula as in (28). The first method, which in four dimensions reads ⁵:

$$\begin{aligned} & \text{sres}_x(\log(D^2)) \\ &= \text{sres}_x(\log_\star(|\xi|^{-2} \star \sigma(D^2))) \\ &+ \frac{1}{2} \sum_{j=0}^2 \frac{(-1)^j}{(j+1)!} \text{sres}_x \left(\left(\log |\xi|^2 \star (|\xi|^{-2} \star \sigma_{<2}(D^2))(x, \xi) \right)^{\star(j+1)} \right)_{<-j-1} \\ &+ \frac{1}{12} \sum_{j=0}^1 \frac{(-1)^j}{(j+1)!} \text{sres}_x \left(\left(\log |\xi|^2 \star \left(\log |\xi|^2 \star (|\xi|^{-2} \star \sigma_{<2}(D^2))(x, \xi) \right)^{\star(j+1)} \right)_{<-j-1} \right)_{<-j-2} \\ &- \frac{1}{12} \text{sres}_x \left(\{ |\xi|^{-2} \star \sigma_{<2}(D^2), (\log |\xi|^2 \star (|\xi|^{-2} \star \sigma_{<2}(D^2))(x, \xi)) \}_{<-1} \star \right), \end{aligned} \quad (47)$$

and therefore requires computing the various terms in the above sums, is lengthier than the second method, which we adopt here.

Replacing the residue by a super residue in (28) yields the following description of the logarithmic superresidue density of D^2 , in which the sum over p reduces to one term.

Proposition 6 *The logarithmic super residue density of the squared Dirac operator is a finite sum of super residues of classical symbols:*

$$\begin{aligned} & \text{sres}_x(\log(D^2)) \\ &= \sum_{|k|=\frac{n}{4}, k_i \in \{1,2\}}^{\frac{n}{2}} \frac{(-1)^{|k|+q-1} (|k|+q-1)!}{k! (k_1+1)(k_1+k_2+2) \cdots (k_1+\cdots+k_p+p)} \\ &\times \text{sres}_x \left((L_x + \Delta_x)^{k_1} (\sigma_{<2}(D^2)) \cdots (L_x + \Delta_x)^{k_q} (\sigma_{<2}(D^2)) |\xi|^{-2(|k|+q)} \right) \end{aligned} \quad (48)$$

⁵ As before, here $\sigma_{<k}$ stands for the part of the symbol σ of order smaller than k .

where as before, we have set $q = \frac{n}{4}$. It is of the form:

$$\text{sres}_x(\log(D^2)) = \sum_{s+t=q} \alpha_{s,t} \text{sres}_x \left((L_x^2 \sigma_{<2}(D^2))^s (\Delta_x \sigma_{<2}(D^2))^t |\xi|^{-2(3s+2t)} \right)$$

with $\Delta_x \sigma_{<2}(D^2)$ and $L_x^2 \sigma_{<2}(D^2)$ contributing respectively by

$$\Delta_x (\Gamma_{ij}^k \Gamma_{lm}^n) \sigma_{jk} \sigma_{mn} = -\frac{1}{2} R_{jkia} R_{nmia} \sigma_{kj} \sigma_{nm} \quad (49)$$

and

$$L_x^2 (\Gamma_{ij}^k \Gamma_{lm}^n) \sigma_{jk} \sigma_{mn} = -R_{jkia} R_{mnib} \sigma_{jk} \sigma_{mn} \xi_a \xi_b. \quad (50)$$

Proof: Applying (28) to $Q = D^2$ yields an expression which involves terms $(L_x + \Delta_x)^{k_i}(\sigma_{<2}(D^2))$, each of which differentiates $\sigma_{<2}(D^2)$ at least k_i times. We have $k_i \leq 2$; indeed, $\text{sres}_x(\log_\star(D^2))$ being proportional to a Pontryagin form, it only involves curvature terms so that only first order derivatives of the Christoffel symbols can arise. In view of the product term $\Gamma_{ij}^k \Gamma_{im}^n$, it can involve at most partial differential operators of order two.

Since the superresidue density $\text{sres}_x(\log D^2) dx$ is proportional to a Pontryagin form, there is no contribution from the scalar curvature, so that terms $\sigma_{<2}(D^2)$ corresponding to zero powers k_i do not contribute.

Let us analyse the contribution of terms involving powers $k_i = 1$ i.e., expressions of the type $(L_x + \Delta_x) \sigma_{<2}(D^2)$. In view of (46) the terms $\Delta_x \sigma_{<2}(D^2)$ can only contribute by

$$\begin{aligned} \Delta_x (\Gamma_{ij}^k \Gamma_{im}^n) \sigma_{kj} \sigma_{nm} &= -2 \partial_a \Gamma_{ij}^k \partial_a \Gamma_{im}^n \sigma_{kj} \sigma_{nm} \\ &= -\frac{1}{2} R_{jkia} R_{nmia} \sigma_{kj} \sigma_{nm}. \end{aligned} \quad (51)$$

Let us now see how the terms $L_x \sigma_{<2}(D^2)$ contribute. We have

$$L_x \sigma_{<2}(D^2) = -2i (\partial_a \Gamma_{ij}^k \sigma_{kj} \xi_i \xi_a + \partial_a \partial_i \Gamma_{ij}^k \sigma_{kj} \xi_a + (\partial_a \Gamma_{ij}^k \Gamma_{lm}^n + \Gamma_{ij}^k \partial_a \Gamma_{lm}^n) \sigma_{kj} \sigma_{nm} \xi_a + \partial_a s \xi_a),$$

which at the center of a normal coordinate system reads:

$$L_x \sigma_{<2}(D^2) = -2i (\partial_a \Gamma_{ij}^k \sigma_{kj} \xi_i \xi_a + \partial_a \partial_i \Gamma_{ij}^k \sigma_{kj} \xi_a + \partial_a s \xi_a).$$

The only possible contribution can come from

$$L_x (\Gamma_{ij}^k \sigma_{kj} \xi_i) = -2i \partial_a \Gamma_{ij}^k \sigma_{kj} \xi_i \xi_a = -i R_{jkia} \sigma_{kj} \xi_i \xi_a, \quad (52)$$

which vanishes by antisymmetry of R . There is therefore no contribution from terms of the type $L_x \sigma_{<2}(D^2)$.

When $k_i = 2$, i.e., terms

$$(L_x + \Delta_x)^2 \sigma_{<2}(D^2) = L_x^2 \sigma_{<2}(D^2) + 2L_x \Delta_x \sigma_{<2}(D^2) + \Delta_x^2 \sigma_{<2}(D^2)$$

only contribute by $L_x^2 \sigma_{<2}(D^2)$, which introduces terms of the type

$$\begin{aligned} L_x^2 (\Gamma_{ij}^k \Gamma_{im}^n \sigma_{kj} \sigma_{nm}) &= -4 \partial_a \Gamma_{ij}^k \partial_b \Gamma_{im}^n \sigma_{kj} \sigma_{nm} \xi_a \xi_b \\ &= -R_{jkia} R_{mnib} \sigma_{jk} \sigma_{mn} \xi_a \xi_b. \end{aligned} \quad (53)$$

To sum up we only have contributions from $\Delta_x \sigma_{<2}(D^2)$ and $L_x^2 \sigma_{<2}(D^2)$ via products

$$(L_x^2 \sigma_{<2}(D^2))^s (\Delta_x \sigma_{<2}(D^2))^t \quad \text{with } p = s + t \quad \text{and} \quad |k| = 2s + t.$$

Since the residue picks the $-n$ -th power in $|\xi|$, we have

$$2s - 2(|k| + q) = -n \implies 2s + 2t = \frac{n}{2} \implies q = \frac{n}{4}.$$

This is confirmed by counting the Clifford coefficients since (3) implies that $2q = 2s + 2t = \frac{n}{2}$.

Combining this with (51) and (53) leads to the statement of the proposition. \square

Example 9 When $n = 4$, in which case $q = 1$, we have $s + t = 1$ so that we need to consider two types of terms: $\Delta_x \sigma_{<2}(D^2)$ and $L_x^2(\sigma_{<2}(D^2))$.

Proposition 6 combined with Proposition 5 yields

$$\begin{aligned} & \text{sres}_x(\log(D^2)) \\ &= -\frac{1}{2} \text{sres}_x(\Delta_x(\sigma_{<2}(D^2)) |\xi|^{-4}) \quad (s = 0, t = 1, k_1 = 1) \\ &+ \frac{\text{sres}_x(L_x^2(\sigma_{<2}(D^2)) |\xi|^{-6})}{2 \times 3} \quad (s = 1, t = 0, k_1 = 2) \\ &= \text{sres}_x(|\xi|^{-4} \partial_a \Gamma_{ij}^k \partial_a \Gamma_{im}^n \sigma_{kj} \sigma_{nm}) \quad \text{by (51)} \\ &- \frac{4}{3} \text{sres}_x(|\xi|^{-6} \partial_a \Gamma_{ij}^k \partial_b \Gamma_{im}^n \sigma_{kj} \sigma_{nm} \xi_a \xi_b) \quad \text{by (53)} \\ &= \left(1 - \frac{1}{3}\right) \text{sres}_x(|\xi|^{-4} \partial_a \Gamma_{ij}^k \partial_a \Gamma_{im}^n \sigma_{kj} \sigma_{nm}) \quad \text{by (43)} \\ &= \frac{1}{48 \pi^2} \text{tr}(R \wedge R) \quad \text{by (42)}. \end{aligned}$$

Once integrated over the manifold M this yields back the well known formula:

$$\text{ind}(D) = \int_M \hat{A} \implies \text{sres}(\log(D^2)) = -2 \int_M \hat{A} = \frac{1}{48 \pi^2} \text{tr}(R \wedge R), \quad (54)$$

since $\hat{A} = 1 - \frac{1}{24(2\pi)^2} \text{tr}(R \wedge R) + \dots$.

Appendix: Complex powers and logarithms

An operator $A \in C\ell(M, E)$ has principal angle θ if for every $(x, \xi) \in T^*M - \{0\}$, the leading symbol $(\sigma_A(x, \xi))^L$ has no eigenvalue on the ray $L_\theta = \{re^{i\theta}, r \geq 0\}$; in that case A is elliptic.

Definition 4 We call an operator $A \in C\ell(M, E)$ admissible with spectral cut θ if A has principal angle θ and the spectrum of A does not meet $L_\theta = \{re^{i\theta}, r \geq 0\}$. In particular such an operator is invertible and elliptic. Since the spectrum of A does not meet L_θ , θ is called an Agmon angle of A .

Let $A \in C\ell(M, E)$ be admissible with spectral cut θ and positive order a . For $\text{Re}(z) < 0$, the complex power A_θ^z of A , first introduced by Seeley [Se], is defined by the Cauchy integral

$$A_\theta^z = \frac{i}{2\pi} \int_{\Gamma_{r,\theta}} \lambda_\theta^z (A - \lambda)^{-1} d\lambda, \quad (55)$$

where $\lambda_\theta^z = |\lambda|^z e^{iz(\arg \lambda)}$ with $\theta \leq \arg \lambda < \theta + 2\pi$. In particular, for $z = 0$, we have $A_\theta^0 = I$.

Here

$$\Gamma_{r,\theta} = \Gamma_{r,\theta}^1 \cup \Gamma_{r,\theta}^2 \cup \Gamma_{r,\theta}^3 \quad (56)$$

where

$$\begin{aligned} \Gamma_{r,\theta}^1 &= \{\rho e^{i\theta}, \infty > \rho \geq r\} \\ \Gamma_{r,\theta}^2 &= \{\rho e^{i(\theta-2\pi)}, \infty > \rho \geq r\} \\ \Gamma_{r,\theta}^3 &= \{r e^{it}, \theta - 2\pi \leq t \leq \theta\}, \end{aligned}$$

is a contour along the ray L_θ around the non zero spectrum of A . Here r is any small positive real number such that $\Gamma_{r,\theta} \cap Sp(A) = \emptyset$.

The definition of complex powers can be extended to the whole complex plane by setting $A_\theta^z := A^k A_\theta^{z-k}$ for $k \in \mathbb{N}$ and $\operatorname{Re}(z) < k$; this definition is independent of the choice of k in \mathbb{N} and preserves the usual properties, i.e. $A_\theta^{z_1} A_\theta^{z_2} = A_\theta^{z_1+z_2}$, $A_\theta^k = A^k$, for $k \in \mathbb{Z}$.

Given an admissible operator A in $C\ell(M, E)$ with zero order and spectral cut θ , its complex powers give rise to a holomorphic map $z \mapsto A_\theta^z$ on the complex plane with values in $\mathcal{B}(H^s(M, E))$ for any real number s , where $H^s(M, E)$ stands for the H^s -closure of the space $C^\infty(M, E)$ of smooth sections of E (see e.g. [G]). The logarithm of A is the bounded operator on $H^s(M, E)$ defined in terms of the derivative at $z = 0$ of this complex power:

$$\begin{aligned} \log_\theta A &:= (\partial_z A_\theta^z)|_{z=0} \\ &= \frac{i}{2\pi} \left(\partial_z \int_{\Gamma_{r,\theta}} \lambda_\theta^z (A - \lambda I)^{-z} d\lambda \right) \Big|_{z=0} \\ &= \frac{i}{2\pi} \int_{\Gamma_{r,\theta}} \log_\theta \lambda (A - \lambda I)^{-z} d\lambda \end{aligned}$$

with the notation of (55).

The notion of logarithm extends to an admissible operator A with positive order a and spectral cut θ in the following way. For any positive ϵ , the map $z \mapsto A_\theta^{z-\epsilon}$ of order $a(z-\epsilon)$ defines a holomorphic function on the half plane $\operatorname{Re}(z) < \epsilon$ with values in $\mathcal{B}(H^s(M, E))$ for any real number s . Thus we can set

$$\log_\theta A = A_\theta^\epsilon (\partial_z (A_\theta^z - \epsilon))|_{z=0} = A_\theta^\epsilon \left(\partial_z \left(\frac{i}{2\pi} \int_{\Gamma_{r,\theta}} \lambda_\theta^{z-\epsilon} (A - \lambda)^{-1} d\lambda \right) \right) \Big|_{z=0}. \quad (57)$$

for any positive ϵ the operator $\log_\theta A A^{-\epsilon} = A^{-\epsilon} \log_\theta A$ lies in $\mathcal{B}(H^s(M, E))$ for any real number s . It follows that $\log_\theta A$, which is clearly independent of the choice of $\epsilon > 0$, defines a bounded linear operator from $H^s(M, E)$ to $H^{s-\epsilon}(M, E)$ for any positive ϵ . We have

$$\sigma(\log_\theta A) = a \log |\xi| + \sigma_0(\log_\theta A)$$

where $\sigma_0(\log_\theta A)$ is a classical symbol whose asymptotic expansion

$$\sigma_0(\log_\theta A) \sim \sum_{j=0}^{\infty} \sigma_{a-j,0}(\log_\theta A)$$

has homogeneous components of the form

$$\sigma_{-j,0}(\log_\theta A)(x,\xi) = |\xi|^{-j} \partial_z \left(\sigma(A_\theta^z)_{\alpha(z)-j} \left(x, \frac{\xi}{|\xi|} \right) \right) \Big|_{z=0}. \quad (58)$$

References

- [ABP] M. Atiyah, R. Bott, V. Patodi, *On the heat equation and the index theorem*, Inv. Math. **19** 279-330 (1973)
- [AG] L. Alvarez-Gaume, *Supersymmetry and the Atiyah-index theorem*, Comm. Math. Phys. **90** 161-173 (1983)
- [AS] M. Atiyah, I.M. Singer, *The index of elliptic operators III*, Ann. Math. Studies 546-604 (1968)
- [BGV] N. Berline, E. Getzler, M. Vergne, **Heat kernels and Dirac operators. Grundlehren der Mathematischen Wissenschaften 298**, Springer Verlag, Berlin 1992.
- [G] P. Gilkey, **Invariance theory, the heat equation and the Atiyah-Singer index theorem**, Second Edition, Studies in advanced mathematics 1995
- [Ge1] E. Getzler, *Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem*, Comm. Math. Phys. **92** 163-178 (1983)
- [Ge2] E. Getzler, *A short proof of the local Atiyah-Singer index theorem*, Topology **25** 111-117 (1986)
- [Gu] V. Guillemin, *Gauged Lagrangian distributions*, Adv. Math. **102** n. 2 184-201 (1993)
- [Ka] Ch. Kassel, *Le résidu non commutatif (d’après M. Wodzicki)*, Séminaire Bourbaki, Astérisque **177-178** (1989) 199–229
- [KV] M. Kontsevich, S. Vishik, *Geometry of determinants of elliptic operators*, Func. Anal. on the Eve of the XXI century, Vol I, Progress in Mathematics **131** (1994) 173–197 ; *Determinants of elliptic pseudodifferential operators*, Max Planck Preprint (1994)
- [LM] H. B. Lawson, M.-L. Michelson, **Spin geometry**, Princeton University Press, 1989
- [MS] H.P. Mc Kean and I.M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geom. **1** (1967), 43–69
- [MP] J. Mickelsson, S. Paycha, *Renormalised Chern-Weil forms associated with families of Dirac operators*, J. Geom. Phys. Vol **57** (2007)
- [O1] K. Okikiolu, *The Campbell-Hausdorff theorem for elliptic operators and a related trace formula*, Duke. Math. Journ. **79** (1995) 687–722
- [O2] K. Okikiolu, *The multiplicative anomaly for determinants of elliptic operators*, Duke Math. Journ. **79** (1995) 722–749
- [P] S. Paycha, *Noncommutative formal Taylor expansions and second quantised regularised traces*, Clay Mathematics Institute proceedings “Combinatorics and Physics”

- [PS] S. Paycha, S. Scott, *A Laurent expansion for regularised integrals of holomorphic symbols*, Geom. Funct. Anal., Geom. Funct. Anal. **17** (2007) 491-536
- [Q] D. Quillen, *Superconnections and the Chern character*, Topology **24** (1985) 89–95
- [Sc1] S. Scott, *The residue determinant*, Commun. Part. Diff. Eqn.s **30** no. 4-6 (2005) 483–507
- [Sc2] S. Scott, *Logarithmic structures and TQFT*, Preprint 2010
- [Sc3] S. Scott, *Traces and determinants of pseudodifferential operators*, Oxford University Press, Math. Monographs, to appear
- [Se] R.T. Seeley, *Complex powers of an elliptic operator, Singular integrals*, Proc. Symp. Pure Math., Chicago, Amer. Math. Soc., Providence (1966) 288–307
- [W] M. Wodzicki, *Non commutative residue. Chapter I. Fundamentals* in Lecture Notes in Math. **1289** 320-399, Springer Verlag 1987; *Spectral asymmetry and noncommutative residue* (in Russian) Thesis, (former) Steklov Institute, Sov. Acad. Sci. Moscow 1984